Problem Set

Solve these problems without using internet resources or asking a colleague.
Submit your answers **typed (not handwritten)** as a pdf document. Use free tools for latex like overleaf.com if you do not have suitable software on your computer. Knowledge of latex is required for this position.

Question # 1

Give a formal proof that if a function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is $f(x) > 0$ for any x then

$$\arg\max_x f(x) = \arg\max_x \log f(x). \tag{1}$$

Hint: It is sufficient to check the necessary conditions on the gradient and on the Hessian for a maximum.

Question # 2

Consider 3D points

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \tag{2}$$

belonging to the hyperplane

$$n_1 x_1 + n_2 x_2 + n_3 x_3 + n_0 = 0. \tag{3}$$

Show that

$$N = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \in \mathbb{R}^3 \tag{4}$$

is a vector normal to the hyperplane. Also, show that $|n_0|/\|N\|$ is the distance of the hyperplane to the origin.

Question # 3

Write the explicit formula of the gradient of

$$E[u] = \sum_{i=2}^{2} \sum_{j=1}^{2} \sin(u[i,j] - u[i-1,j]) + \cos(u[i,j] + u[i-1,j]) \tag{5}$$

with respect to the variable u, which is a 2×2 matrix

$$\begin{bmatrix} u[1,1] & u[1,2] \\ u[2,1] & u[2,2] \end{bmatrix}. \tag{6}$$

Show all the steps of your calculations.

Question # 4

You are given the following probability density distribution

$$p(x; \alpha, \epsilon) = A e^{-\frac{x^2}{\alpha} - 2\epsilon x}, \tag{7}$$

where A is the inverse partition function, $x \in \mathbb{R}^n$, $\alpha > 0$ and ϵ are scalars. Assume that you are given m independent and identically distributed samples $x^{(1)}, \ldots, x^{(m)}$. Let us also assume that ϵ is given. Write the explicit formula of the maximum likelihood estimator for the parameter α. Show all the steps of your calculations.